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Dynamical algebra of spin waves in localised-spin models

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Abstract. The idea of the spectrum generating algebra (SGA) is used to study the low-lying excitations in various localised-spin models of ferromagnetic, ferrimagnetic and antiferromagnetic solids. The noncompact algebra so(2, 1) ~ su(1, 1) is found to generate the spin wave or the magnon spectrum for ferromagnetic, ferrimagnetic and antiferromagnetic Bravais lattices with nearest-neighbour exchange interactions (Heisenberg model). The ferromagnetic case is treated including dipolar interactions in the model. For a quadratic antiferromagnet with easy-plane anisotropy the SGA is found to be the direct product so(2, 1)⊗so(2, 1) ~ so(2, 2). For the XY and the Heisenberg-Ising models of an antiferromagnetic linear chain the SGA is the compact counterpart of so(2, 1), i.e., so(3) ~ su(2). It is concluded that if the magnons are quantised as bosons then the SGA is the noncompact algebra so(2, 1), and if they are quantised as fermions then the SGA is the compact angular momentum algebra so(3).

1. Introduction

It has been known for some time that the energy spectra associated with some Hamiltonians or the eigenvalues of some differential equations can be obtained from a knowledge of suitable representations of certain Lie algebras (Barut 1964a, b, 1973, Lanik 1967, 1968). The algebra used to obtain the eigenvalues and the eigenfunctions is referred to as the spectrum generating algebra (SGA) or the dynamical algebra. Wybourne (1974) and Barut and Raczka (1977) have discussed several interesting applications of the method to familiar quantum mechanical systems. Initial applications of the method were in the study of the mass and energy spectra of elementary particles (Barut 1964a, b, Barut and Böhm 1965, Dothan *et al* 1965) and many applications (Aldrovandi and Leal Ferreira 1969, Cordero and Hojman 1970, Cordero and Ghirardi 1971, Cordero *et al* 1971) involving various single quantum mechanical systems soon followed. The idea was carried over to the realm of second quantised Hamiltonians by Solomon (1971), when he used it to derive the energy of low-lying excitations in a superfluid boson system. This seems to be the first application of the method in the study of a condensed matter system. In a later application Solomon (1981) discussed superconductivity in the BCS model and superfluidity of ^{3}He. Most notable recent applications of the method in the field of condensed matter theory include the study of the charge density wave (CDW) state in one-dimensional many-electron systems (Solomon and Birman 1982), the phenomena of coexistence of the CDW phase and superconductivity (Birman and Solomon 1982) and superconductivity and spin density waves (Solomon and Birman 1984a, b, 1985).

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In view of these developments it is desirable to apply the method to the study of excitations in various magnetic systems. In the present paper we propose to discuss the SGA of spin waves in 'localised-spin' models of magnetic solids. In magnetic crystals some atoms are equipped with magnetic moments which, at sufficiently low temperatures, form a regular pattern with a characteristic unit cell. Spin waves or magnons are the low-lying excitations of this static spin configuration. The classical picture is one where the spin precesses around its equilibrium position at a small angle. The localised-spin models are inadequate to explain the strongly magnetic properties of the iron transition series metals and their alloys, which must be explained by band or the itinerant electron theory of magnetism (see Herring (1966) for a review on this subject). In the present paper we will not consider models of itinerant or partly itinerant (Stearns 1978) electron magnetism but will concentrate entirely on localised-spin models.

The literature on the spin wave theory of localised-spin models is vast and we will refer to the book by Mattis (1981) and an article by Thorpe (1978), which include up-to-date references to various works in this field. Both semiclassical (Heller and Kramers 1934) and quantum mechanical (Slater 1930, Bloch 1932, Holstein and Primakoff 1940) methods have been adopted to study the spin waves. Holstein and Primakoff (1940) used creation and destruction operators to study these magnetic excitations which behave as bosons at sufficiently low temperatures. In an excellent review article, Van Kranendonk and Van Vleck (1958) showed how the spin waves could as well be studied as harmonic oscillator states. In their words 'the distinction (of the harmonic oscillator approach) from the conventional approach of Bloch (1932), and of Holstein and Primakoff (1940), based on creation and annihilation operators, is to a considerable extent only a semantic one, but nevertheless is probably of use to those readers to whom harmonic oscillators are more intuitive than the techniques of quantum field theory'. This idea that the spin waves in various localised-spin models can be viewed as harmonic oscillator states is important for the study of these excitations from a group-theoretical or algebraic point of view. The harmonic oscillator spectrum is discrete, integer-spaced, with a lower bound and without an upper bound. This spectrum is identical with the spectrum of one of the generators J_1 of the noncompact Lie algebra so(2, 1) in the D^+ representation (Wybourne 1974, pp 146-8), where J_1 is also the generator of the corresponding compact subalgebra so(2). Indeed, so(2, 1) is known as the SGA of an isotropic harmonic oscillator (Wybourne 1974, ch 18). Consequently spin waves can be visualised as either boson excitations or 'harmonic oscillator' states of a localised-spin magnetic system or, from an algebraic point of view, as the eigenstates of one of the generators of so(2, 1). While the difference between the former two viewpoints is largely a semantic one, the latter viewpoint may provide further insight into the study of more complex phenomena such as the interplay and coexistence of superconductivity, magnetism and CDW (Huang 1985). The purpose of the present paper, apart from offering this alternative viewpoint, is to unify the results obtained for models of ferromagnetic, ferrimagnetic and antiferromagnetic materials and to point out the similarity in the algebraic structure of the Hamiltonians in these models. In most models this algebraic structure is that of so(2, 1) or the isomorphic algebra su(1, 1). When the spectrum shows two distinct branches, as in a quadratic antiferromagnet with an easy-plane anisotropy (Lebesque *et al* 1978, Balucani *et al* 1980), one has to consider the direct product so(2, 1)$\otimes$so(2, 1), with each algebra giving rise to one branch of the spectrum. In the XY and Heisenberg-Ising models of an antiferromagnetic chain (Lieb *et al* 1961) the spin wave excitations appear to

obey the fermion statistics. In such a case the dynamical algebra is the compact counterpart of so(2, 1), i.e., so(3) ~ su(2). Thus it is expected that if the magnons are quantised as bosons then the SGA is so(2) ~ su(1, 1), and if they are quantised as fermions then the SGA is so(3) ~ su(2). In the appendix we provide arguments in favour of this conjecture.

In our discussion of the dynamical algebra of various cases we will use Hamiltonians expressed in harmonic oscillator coordinates as well as second quantised operators according to our convenience. This will help us use previously derived results as well as make contact with both the approaches. The present paper is divided into the following sections. In § 2 we provide a brief outline of the method of dynamical algebra. In §§ 3-6 we apply the method to various localised-spin models of magnetic solids. In § 7 we comment on the directions in which future work in this field should be done.

2. The method of dynamical algebra

The method of dynamical or spectrum generating algebra has been adequately discussed by Wybourne (1974, ch 18) and Solomon (1981). In this method the Hamiltonian of an interacting system is first expressed as an element of a Lie algebra g. This algebra is determined by the commutation relations between the operators appearing in the Hamiltonian. If the Hamiltonian is given by a sum over a particular quantum label k, then g is the direct sum of isomorphic Lie algebras g_k:

$$g = \sum_k g_k,$$

and algebraic treatment of the Hamiltonian is determined essentially by g_k. The spectrum and the eigenstates of the Hamiltonian are generated in a suitable unitary representation of g_k, the need for a unitary representation arising because of the hermiticity of the Hamiltonian.

The Hamiltonian is first written as a linear combination of the generators of the algebra g_k. The next step is the diagonalisation of the Hamiltonian. This is achieved by an automorphism of g_k, which can be brought about by subjecting the Hamiltonian to a suitable rotation in the space of g_k. The rotation is equivalent to a Bogoliubov transformation and is chosen so that the Hamiltonian can be written as a sum of commuting generators of g_k. The spectrum of these generators in a suitable unitary representation of g_k provides the spectrum of the Hamiltonian. The energy eigenstates can be obtained from a knowledge of this rotation and the eigenstates of the diagonalising generators.

It has been pointed out by Solomon (1981) that the rotation needed to diagonalise the Hamiltonian need not be carried out explicitly. Instead, the Hamiltonian can be expressed as a matrix M in a low-dimensional faithful representation of the Lie algebra. One can then make use of the invariants

$$\operatorname{Tr} M^n, \qquad n = 1, 2, \ldots, \tag{1}$$

of which there are l independent ones for a Lie algebra of rank l. In our study of the dynamical algebra of spin waves we will have the opportunity to illustrate this method as well as the one involving rotation.

3. Spin waves in a ferromagnet

We start by considering the simplest model of a ferromagnetic system that exhibits spin waves. We consider a system of spins localised on a Bravais lattice placed in an external magnetic field, and assume a simple isotropic Heisenberg interaction between the nearest neighbours. The Hamiltonian of this spin system is

$$H = -\tfrac{1}{2}\sum_{ij} J_{ij}\boldsymbol{S}_i\cdot\boldsymbol{S}_j - h\sum_i S_i^z. \tag{2}$$

In (2), h is an external field dependent quantity. The exchange integral J_{ij} is equal to J when i and j are nearest neighbours to each other and zero otherwise. J is positive for ferromagnetics. The spin–spin interaction term in (2) is called isotropic because the scalar product $\boldsymbol{S}_i\cdot\boldsymbol{S}_j$ depends only on the angle between the vectors $\boldsymbol{S}_i$ and $\boldsymbol{S}_j$ and is independent of the direction of their resultant.

The low-lying excitations of the Hamiltonian (2) are spin waves. If one neglects interactions among the spin waves, then this Hamiltonian can be represented in a diagonal form in boson creation and annihilation operators (Holstein and Primakoff 1940) or as a sum over harmonic oscillator Hamiltonians (Van Kranendonk and Van Vleck 1958). To study the dynamical algebra of (2) one could use either form. In the present section we will use the harmonic oscillator form of the Hamiltonian (2).

To achieve the transformation of (2) to harmonic oscillator coordinates, one first considers the spin-deviation quantum number $n_i = s - m$, where m is the z component of the atomic spin $\boldsymbol{S}_i$. n_i measures the deviation of S_i^z from its maximum value s. The eigenvalues of the spin-deviation operator $n_i = s - S_i^z$ are $0, 1, 2, \ldots, 2s$. In a representation in which this operator is diagonal, the nonvanishing matrix elements of the three components of $\boldsymbol{S}_i$ are given by

$$\begin{aligned}
&\langle n_i|S_i^x|n_i+1\rangle = \langle n_i+1|S_i^x|n_i\rangle = \tfrac{1}{2}(n_i+1)^{1/2}(2s-n_i)^{1/2},\\
&\langle n_i|S_i^y|n_i+1\rangle = \langle n_i+1|S_i^y|n_i\rangle^* = -\tfrac{1}{2}\mathrm{i}(n_i+1)^{1/2}(2s-n_i)^{1/2},\\
&\langle n_i|S_i^z|n_i\rangle = s-n_i, \qquad 0\leqslant n_i\leqslant 2s.
\end{aligned} \tag{3}$$

The matrices of the components of $\boldsymbol{S}_i$ have the form (3) with respect to n_i, but are diagonal with respect to n_i of all the other atoms in the crystal. Thus the matrices of $\boldsymbol{S}_i$ are the direct products of the matrices (3) and $N-1$ unit matrices, where N is the total number of spins.

We now consider the matrix elements of the coordinate x_i and the momentum p_i of a harmonic oscillator of mass m and angular frequency ω between its energy eigenstates $|n_i\rangle$ (see e.g. Gottfried 1974):

$$\begin{aligned}
&\langle n_i|x_i|n_i+1\rangle = \langle n_i+1|x_i|n_i\rangle = (\hbar/2m\omega)^{1/2}(n_i+1)^{1/2},\\
&\langle n_i|p_i|n_i+1\rangle = \langle n_i+1|p_i|n_i\rangle^* = -\mathrm{i}(\hbar m\omega/2)^{1/2}(n_i+1)^{1/2},
\end{aligned} \tag{4}$$

where n_i corresponds to the number of quanta with which the oscillator is excited, i.e.,

$$\langle n_i|p_i^2/2m + (m\omega^2/2)x_i^2 - \hbar\omega/2|n_i\rangle = n_i\hbar\omega. \tag{5}$$

Introducing dimensionless variables

$$Q_i = (m\omega/\hbar)^{1/2}x_i, \qquad P_i = (\hbar m\omega)^{-1/2}p_i, \tag{6}$$

equations (3) and (4) can be written as

$$\begin{aligned}
&\langle n_i|s^{1/2}Q_i|n_i+1\rangle = \langle n_i+1|s^{1/2}Q_i|n_i\rangle = \tfrac{1}{2}(n_i+1)^{1/2}(2s)^{1/2},\\
&\langle n_i|s^{1/2}P_i|n_i+1\rangle = \langle n_i+1|s^{1/2}P_i|n_i\rangle^* = -\tfrac{1}{2}\mathrm{i}(n_i+1)^{1/2}(2s)^{1/2},\\
&\langle n_i|s-\tfrac{1}{2}(P_i^2+Q_i^2-1)|n_i\rangle = s-n_i.
\end{aligned} \tag{7}$$

The sets of matrices (3) and (7) are strikingly similar for small values of n_i. The theory developed on the basis of this similarity is valid only for low occupation number for every oscillator. For $n_i \approx s$, the approximation is quantitatively incorrect and when $n_i \geqslant 2s$ it breaks down totally, since although (7) continues to be valid, there is no corresponding structure in angular momentum space.

Thus for small spin-deviations one can make the substitutions

$$S_i^x = Q_i s^{1/2}, \qquad S_i^y = P_i s^{1/2}, \qquad S_i^z = s - \tfrac{1}{2}(P_i^2+Q_i^2-1) \tag{8}$$

in (2) and discarding quartic terms, one obtains the 'linearised' or 'reduced' Hamiltonian

$$H_{\text{red}} = E_0 + (Jsz+h)\sum_i \tfrac{1}{2}(P_i^2+Q_i^2-1) - \tfrac{1}{2}Js\sum_{\text{NN}}(Q_iQ_j+P_iP_j), \tag{9}$$

where z = number of nearest neighbours of any given spin, and

$$E_0 = -hNs - \tfrac{1}{2}NJs^2z$$

is the energy of the completely saturated state in which all spins are parallel to the applied field.

The reduced Hamiltonian (9) represents a set of harmonic oscillators with nearest-neighbour interactions. It can be transformed into a Hamiltonian representing uncoupled oscillators by using the Slater transformation (Slater 1930) to the real standing waves:

$$\left.\begin{aligned} Q_k &= (2/N)^{1/2}\sum_i \cos(\boldsymbol{k}\cdot\boldsymbol{r}_i)Q_i,\\ P_k &= (2/N)^{1/2}\sum_i \sin(\boldsymbol{k}\cdot\boldsymbol{r}_i)P_i, \end{aligned}\right\} \qquad k_x \geqslant 0 \tag{10a}$$

$$\left.\begin{aligned} Q_k &= (2/N)^{1/2}\sum_i \sin(\boldsymbol{k}\cdot\boldsymbol{r}_i)Q_i,\\ P_k &= (2/N)^{1/2}\sum_i \cos(\boldsymbol{k}\cdot\boldsymbol{r}_i)P_i, \end{aligned}\right\} \qquad k_x < 0 \tag{10b}$$

where $\boldsymbol{r}_i$ are the position vectors of the spins in the lattice and $\boldsymbol{k}$ are the reciprocal lattice vectors, assuming periodic boundary conditions for the wavefunctions.

The transformed Hamiltonian is

$$H_{\text{red}} = E_0 + \sum_k \tfrac{1}{2}(P_k^2+Q_k^2-1)\hbar\omega_k, \tag{11}$$

where

$$\hbar\omega_k = h + Js\sum_\gamma (1-\cos\boldsymbol{k}\cdot\boldsymbol{\gamma}) \approx h + \tfrac{1}{2}Jsk^2a^2 + \mathrm{O}(k^4). \tag{12}$$

The sum in (12) is over the vectors $\boldsymbol{\gamma}$ connecting a given spin to its z nearest neighbours and $a = |\boldsymbol{\gamma}|$ for a simple cubic lattice. $\hbar\omega_k$ is interpreted as the energy of the 'magnons' or 'spin waves' in state $\boldsymbol{k}$. The eigenvalues of the operator $\frac{1}{2}(P_k^2+Q_k^2-1)$ are integers $0, 1, 2, \ldots$ and are interpreted as the number of magnons in the state $\boldsymbol{k}$. This result follows from the known integer-spaced energy spectrum of a harmonic oscillator.

Alternatively one can define boson creation and annihilation operators (see e.g. the analogous discussion of a phonon spectrum by Kittel (1963, ch 2)) by taking linear combinations of the momentum and the position operators in (11) and express the Hamiltonian in a bilinear form in these operators, the latter being interpreted as the occupation number operator. This, in effect, relates the harmonic oscillator approach to that of Holstein and Primakoff (1940).

Alternative to the harmonic oscillator and the boson excitation approach is the algebraic method, where one visualises the Hamiltonian (11) as an element of a Lie algebra. As mentioned in § 1, the operator $\frac{1}{2}(P_k^2+Q_k^2)$ can be expressed as one of the generators J_1^k of the noncompact algebra so(2, 1), where J_1^k is also the generator of the compact subalgebra so(2). To see this explicitly we define operators J_1^k and J_2^k as

$$J_1^k = P_k^2 + Q_k^2/4, \tag{13}$$

$$J_2^k = P_k^2 - Q_k^2/4. \tag{14}$$

By using the commutation relation

$$[Q_k, P_{k'}] = \mathrm{i}\delta_{kk'},$$

we find

$$[J_1^k, J_2^k] = \tfrac{1}{2}\mathrm{i}(Q_kP_k - \mathrm{i}/2). \tag{15}$$

The algebra generated by the commutation of J_1^k and J_2^k closes upon the introduction of only one additional operator

$$J_3^k = \tfrac{1}{2}(Q_kP_k - \mathrm{i}/2), \tag{16}$$

and the Hermitian operators J_1^k, J_2^k and J_3^k have the commutation rules

$$[J_1^k, J_2^k] = \mathrm{i}J_3^k, \qquad [J_2^k, J_3^k] = -\mathrm{i}J_1^k, \qquad [J_3^k, J_1^k] = \mathrm{i}J_2^k. \tag{17}$$

If the second commutator had a positive sign, this would correspond to the well known three-dimensional angular momentum algebra so(3) or the isomorphic algebra su(2). The angular momentum algebra so(3) or the group SO(3) is compact, since its parametrisation consists of a finite number (three in this case) of bounded domains. The negative sign in (17) makes two of the parameter domains of the corresponding algebra unbounded. This algebra, denoted by so(2, 1), is therefore called noncompact. A result of the noncompactness is that all unitary representations (requiring $J_i^+ = J_i$) of so(2, 1) are infinite dimensional (the unitary representations of the compact algebra so(3) are finite dimensional).

These unitary representations can be labelled by the simultaneous eigenvalues of one of the generators J_i and the Casimir operator J^2 which commutes with all elements of the algebra. The Casimir operator for so(2, 1) is given by

$$J_k^2 = (J_2^k)^2 + (J_3^k)^2 - (J_1^k)^2. \tag{18}$$

If one diagonalises J^2 along with the generator J_1 of the compact subalgebra so(2), then the corresponding representation may be divided into two distinct categories: C(continuous) and D(discrete). In the C representation the eigenvalues of J^2 are continuous, with a lower bound but without an upper bound, while the eigenvalues of J_1 are discrete but unbounded both above and below. Since the spectrum of the Hamiltonian representing a physical system must be bounded below, the C representation is not suitable for describing the spectrum of such a Hamiltonian. In the D

representations the eigenvalues of J^2 are discrete and the eigenvalues of J_1 form infinite, integer-spaced spectra. In one of these discrete representations, denoted usually by D_j^+ (Wybourne 1974, pp 146-8), the spectrum of J_1 is bounded below:

$$J^2 = |ja\rangle = j(1-j)|ja\rangle, \qquad j = \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots \tag{19}$$

and

$$J_1|ja\rangle = a|ja\rangle, \tag{20}$$

where

$$a = j + n, \qquad n = 0, 1, 2, \ldots .$$

The spectrum (20) with $j = \frac{1}{2}$ is the harmonic oscillator spectrum and will be used repeatedly in discussing the dynamical algebra of spin waves.

Instead of diagonalising J^2 and J_1 simultaneously, one may choose to diagonalise J^2 along with one of the generators of the noncompact subalgebra so(1, 1), i.e., J_2 or J_3. The spectrum of J_2 or J_3 in the resulting unitary representation is continuous and bounded below (Barut and Phillips 1968).

We now return to the Hamiltonian (11). In terms of the generators of so(2, 1) it can be written as

$$H_{\text{red}} = E_0 + \sum_k H_k, \qquad H_k = (2J_1^k - \tfrac{1}{2})\hbar\omega_k, \tag{21}$$

i.e. the Hamiltonian is an element of the direct sum of isomorphic Lie algebras $\text{so}(2, 1)_k$. The eigenvalues of H_{red} are obtained from the sum of the eigenvalues of J_1^k and its eigenvectors are the products of the eigenvectors of J_1^k. To obtain these, we need the eigenvalues of the Casimir operator (18). With the definitions (13), (14) and (16), we find

$$J^2 = \tfrac{3}{16}, \qquad \text{i.e., } j = \tfrac{3}{4}, \tfrac{1}{4}. \tag{22}$$

In the representation $D_{3/4}^+$ of so(2, 1)

$$J_1^k|\tfrac{3}{4}, \tfrac{3}{4} + n\rangle = (\tfrac{3}{4} + n)|\tfrac{3}{4}, \tfrac{3}{4} + n\rangle, \tag{23}$$

and in the representation $D_{1/4}^+$

$$J_1^k|\tfrac{1}{4}, \tfrac{1}{4} + n\rangle = (\tfrac{1}{4} + n)|\tfrac{1}{4}, \tfrac{1}{4} + n\rangle, \qquad n = 0, 1, 2, \ldots . \tag{24}$$

Thus

$$H_k|\tfrac{3}{4}, \tfrac{3}{4} + n\rangle = (2n+1)\hbar\omega_k|\tfrac{3}{4}, \tfrac{3}{4} + n\rangle, \tag{25}$$

$$H_k|\tfrac{1}{4}, \tfrac{1}{4} + n\rangle = 2n\hbar\omega_k|\tfrac{1}{4}, \tfrac{1}{4} + n\rangle. \tag{26}$$

Thus the harmonic oscillator eigenstates split into two representations of the algebra $\text{so}(2, 1) \sim \text{su}(1, 1)$ according to whether the oscillator energy level is even or odd. These are, however, not the 'true' representations of so(2, 1) or the isomorphic algebra su(1, 1). In a 'true' representation of this algebra, j is an integer or a half integer (Ui 1968) as in equation (19), since it is the lowest eigenvalue of J_1, the generator of the corresponding angular momentum subalgebra so(2). The 'one-boson system' typifies a fractional angular momentum! A partial realisation of the harmonic oscillator states in a 'true' representation of so(2, 1) is discussed in the appendix.

In view of the above discussion we conclude that the spectrum generating algebra of the original Hamiltonian (2) is the direct product $\Pi_k \otimes \text{so}(2, 1)_k$. Energy eigenstates

of (2) with even and odd numbers of magnons are realised in the representations $D^+_{1/4}$ and $D^+_{3/4}$ of so(2, 1) respectively. This result is equally applicable to the spectrum of noninteracting phonons.

Interactions of dipolar structure can be readily included in the discussion. The interaction term, including those of dipolar origin, can be written as (Van Kranendonk and Van Vleck 1958)

$$H_{\text{int}} = -J \sum_{\text{NN}} \boldsymbol{S}_i \cdot \boldsymbol{S}_j + \sum_{j>i} D_{ij}[\boldsymbol{S}_i \cdot \boldsymbol{S}_j - 3(\boldsymbol{\alpha}_{ij} \cdot \boldsymbol{S}_i)(\boldsymbol{\alpha}_{ij} \cdot \boldsymbol{S}_j)]. \tag{27}$$

The summation over i and j in the second term has to be extended over all pairs of spins in the lattice rather than only over the nearest neighbours. The components of the unit vector $\boldsymbol{\alpha}_{ij}$ are the direction cosines of the vector $\boldsymbol{r}_{ij}$ connecting sites i and j. If the dipolar interaction has electromagnetic origin, then $D_{ij} \sim 1/r^3_{ij}$. However, contributions to D_{ij} can also arise from anisotropic exchange and the resulting values of D_{ij} for adjacent atoms can be appreciably greater than those given by classical electromagnetic theory.

The transformation of (27) to harmonic oscillator variables proceeds as before (see Mattis 1981, Van Kranendonk and Van Vleck 1958 for details). The resulting Hamiltonian contains terms up to quartic order in the harmonic oscillator variables. The term independent of those variables provides an additive constant to (27) and can be dropped. The cubic and the quartic terms represent anharmonic corrections and give rise to interactions among the spin waves. Thus they can be dropped if one is interested only in the spectrum of noninteracting excitations. The contribution due to the linear term is a factor $N^{1/2}$ smaller than that due to the quadratic term, and hence can be dropped in the limit of large N. Thus, retaining only the quadratic terms, one obtains the form

$$H_{\text{red}} = \sum_k \tfrac{1}{2}[A(\boldsymbol{k})Q^2_k + B(\boldsymbol{k})P^2_k + 2C(\boldsymbol{k})Q_kP_k] + E'_0, \tag{28}$$

where E'_0 is a constant. The quantities A, B and C are given by lattice Fourier sums involving D_{ij} and the components of the vector $\boldsymbol{\alpha}_{ij}$. In the absence of the dipolar interaction, C vanishes, and A and B reduce to the energy of a spin wave k given by (12). A, B and C satisfy the conditions $A>0$, $B>0$; $A, B>C$. The Hamiltonian (28) can be diagonalised by Bogoliubov transformation to a new set of harmonic oscillator coordinates. However, this transformation is equivalent to a rotation in the space of the algebra so(2, 1). In terms of the generators J^k_i $(i=1,2,3)$, we have

$$H_{\text{red}} - E'_0 = \sum_k H_k$$

where

$$H_k = [A(\boldsymbol{k}) + B(\boldsymbol{k})]J^k_1 + [A(\boldsymbol{k}) - B(\boldsymbol{k})]J^k_2 + 2C(\boldsymbol{k})J^k_3, \tag{29}$$

where we have included a term involving $C(\boldsymbol{k})$ (c-number) in the zero-point energy E'_0 of the system. We now consider a rotation about the axis J^k_3. Using

$$e^{-iJ_3\theta}J_1\, e^{iJ_3\theta} = J_2 \sinh\theta + J_1 \cosh\theta, \tag{30}$$

$$e^{-iJ_3\theta}J_2\, e^{iJ_3\theta} = J_2 \cosh\theta + J_1 \sinh\theta, \tag{31}$$

we obtain the rotated Hamiltonian

$$H'_k = e^{-iJ_3\theta}H_k\, e^{iJ_3\theta} = J^k_1[4A(\boldsymbol{k})B(\boldsymbol{k})]^{1/2} + 2C(\boldsymbol{k})J^k_3, \tag{32}$$

where we have chosen θ to satisfy

$$\tanh\theta=[B(\boldsymbol{k})-A(\boldsymbol{k})]/[B(\boldsymbol{k})+A(\boldsymbol{k})]. \tag{33}$$

A further rotation of H'_k about the axis J_2^k, utilising

$$e^{-i\phi J_2}J_1\,e^{i\phi J_2}=J_1\cosh\phi-J_3\sinh\phi, \tag{34}$$

$$e^{-i\phi J_2}J_3\,e^{i\phi J_2}=J_3\cosh\phi-J_1\sinh\phi, \tag{35}$$

and choosing ϕ to satisfy

$$\tanh\phi=C(\boldsymbol{k})/(A(\boldsymbol{k})B(\boldsymbol{k}))^{1/2}, \tag{36}$$

yields

$$H''_k=\exp(-i\phi J_2^k)H'_k\exp(i\phi J_2^k)=2J_1^k[A(\boldsymbol{k})B(\boldsymbol{k})-C(\boldsymbol{k})^2]^{1/2}. \tag{37}$$

This form is similar to (21) and all the previously derived results can be used to discuss the spectrum. The rotations needed to eliminate J_1^k from H_k cannot be realised since $A(k)+B(k)>A(k)-B(k)$ and $A(k)B(k)>C(k)^2$. This eliminates the possibility of a continuous spectrum for H_k, which would require that H_k be expressed only in terms of the generator J_2^k or J_3^k of the noncompact subalgebra so(1, 1).

The eigenvalue equation

$$H''_k|n_k\rangle=E_k|n_k\rangle$$

is equivalent to

$$H_k\exp(i\theta J_3^k)\exp(i\phi J_2^k)|n_k\rangle=E_k\exp(i\theta J_3^k)\exp(i\phi J_2^k)|n_k\rangle, \tag{38}$$

i.e., the eigenvalues of H_k are those of H''_k, but the eigenstates are to be obtained by subjecting the eigenstate of J_1^k to successive rotations about the axes J_2^k and J_3^k through angles ϕ and θ defined by equations (36) and (33) respectively. The eigenstates of the total Hamiltonian are the products over k of the rotated eigenstates of J_1^k.

As pointed out by Solomon (1981), the above rotation need not be carried out explicitly in order to obtain the spectrum (37). Instead, one can write H_k in a low-dimensional faithful representation of so(2, 1). For example, J_i $(i=1,2,3)$ can be represented by two-dimensional matrices,

$$J_1=\frac{1}{2}\begin{bmatrix}1&\\&-1\end{bmatrix},\qquad J_2=\frac{1}{2}\begin{bmatrix}&1\\-1&\end{bmatrix},\qquad J_3=\frac{-i}{2}\begin{bmatrix}&1\\1&\end{bmatrix}. \tag{39}$$

The Hamiltonian H_k can be written as a matrix M,

$$M=\frac{1}{2}\begin{bmatrix}(A+B)&(A-B)-2Ci\\-(A-B)-2Ci&-(A+B)\end{bmatrix}, \tag{40}$$

where we have suppressed the $\boldsymbol{k}$ dependence of A, B and C for typographical simplicity. We now consider the invariants $\operatorname{Tr}M^n$, $n=1,2,\ldots$. $\operatorname{Tr}M$ is a null invariant and

$$\operatorname{Tr}M^2=2(AB-C^2). \tag{41}$$

Higher values of n do not produce any new invariants for this rank-1 algebra. Thus H_k can be diagonalised to $2J_1^k(AB-C^2)^{1/2}$ if $AB>C^2$ or to $2J_{2,3}^k(C^2-AB)^{1/2}$ if $AB<C^2$. The condition $C^2<AB$ eliminates the latter possibility and hence there is no continuous spectrum associated with H_k.

4. Spin waves in antiferromagnets and ferrimagnets

We now consider a system of $2N$ spins with nearest-neighbour antiferromagnetic exchange interaction in a fictitious magnetic field H_A (Kittel 1963, ch 4). These spins occupy sites on two interpenetrating sublattices a and b such that all the nearest neighbours of an a site are b sites and *vice versa*. The fictitious field H_A approximates the effect of the crystal anisotropy energy. The Hamiltonian for this system is

$$H = J\sum_{j\delta} \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+\delta} - h\sum_j S^a_{jz} + h\sum_j S^b_{jz}, \tag{42}$$

where J is the nearest-neighbour exchange integral, positive for antiferromagnetic interaction. h is a magnetic field (H_A) dependent quantity and the sites j and δ are nearest neighbours of each other. This Hamiltonian can be expressed in harmonic oscillator variables by following the prescription outlined in § 3. The alternative approach is due to Holstein and Primakoff (1940), in which (42) is expressed in boson field operators. Though the present discussion could as well be carried on in the language of harmonic oscillators, we will now switch to the boson excitation picture, thus making explicit contact with both approaches.

In the boson excitation approach one first defines creation and annihilation operators referring to the jth atoms on the sublattices a and b:

$$S^+_{aj} = S^a_{jx} + iS^a_{jy} = (2S)^{1/2}(1 - a_j^+ a_j/2S)^{1/2} a_j, \tag{43a}$$

$$S^-_{aj} = S^a_{jx} - iS^a_{jy} = (2S)^{1/2} a_j^+ (1 - a_j^+ a_j/2S)^{1/2}, \tag{43b}$$

$$S^+_{bl} = (2S)^{1/2} b_l^+ (1 - b_l^+ b_l/2S)^{1/2}, \tag{44a}$$

$$S^-_{bl} = (2S)^{1/2}(1 - b_l^+ b_l/2S) b_l. \tag{44b}$$

S is the magnitude of the atomic spins S_{aj} and S_{bl}. The magnon variables for the sublattices a and b are obtained by plane-wave transformation:

$$C_k = \frac{1}{\sqrt{N}}\sum_j \exp(i\boldsymbol{k}\cdot\boldsymbol{r}_j) a_j, \qquad D_k = \frac{1}{\sqrt{N}}\sum_l \exp(i\boldsymbol{k}\cdot\boldsymbol{r}_l) b_l. \tag{45}$$

The sum for C is over the N atoms j on sublattice a, and for D over the atoms l on b. Neglecting terms higher than quadratic in magnon field operators, the Hamiltonian is reduced to

$$H_{red} = -2NZJS^2 - 2NhS + H_0, \qquad H_0 = \sum_k H_k \tag{46}$$

and

$$H_k = \omega_e \gamma_k (C_k^+ D_k^+ + C_k D_k) + (\omega_e + \omega_A)(C_k^+ C_k + D_k^+ D_k), \tag{47}$$

where

$$\omega_e = 2JZS, \qquad \omega_A = h, \qquad \gamma_k = Z^{-1}\sum_\delta e^{i\boldsymbol{k}\cdot\boldsymbol{\delta}},$$

Z is the number of nearest neighbours and $\boldsymbol{\delta}$ is a vector joining a typical spin to one of its nearest neighbours.

This two-boson Hamiltonian can be expressed as an element of the direct sum of isomorphic Lie algebras so(2, 1) in their 'true' unitary representation (Solomon 1971, Holman and Biedenharn 1966, 1968). Suppressing the superscript k, we define Hermitian

operators J_1, J_2 and J_3:

$$J_1 = -(C^+D^+ + CD)/2, \tag{48}$$

$$J_2 = \mathrm{i}(C^+D^+ - CD)/2, \tag{49}$$

$$J_3 = (C^+C + D^+D + 1)/2. \tag{50}$$

These obey the commutation rules of $\mathrm{so}(2,1) \sim \mathrm{su}(1,1)$:

$$[J_1, J_2] = -\mathrm{i}J_3, \qquad [J_2, J_3] = \mathrm{i}J_1, \qquad [J_3, J_1] = \mathrm{i}J_2.$$

This makes H_k a linear combination of J_3 and J_1, and a rotation about J_2 by an angle θ, given by

$$\tanh\theta = \omega_e\gamma_k/(\omega_e + \omega_A),$$

may be used to eliminate J_1. Since $\gamma_k < 1$, the rotation needed to eliminate the compact generator J_3 would require $\tanh\theta > 1$. This rules out the possibility of a continuous spectrum for H_k.

The Casimir operator in this case has the value

$$J^2 = j(1-j) = \tfrac{1}{4}(1-\Delta^2), \tag{51}$$

where

$$\Delta = |C^+C - D^+D|, \tag{52}$$

$$j = (1+\Delta)/2. \tag{53}$$

Δ is an integer, being the difference between the eigenvalues of two boson number operators, and its lowest eigenvalue is zero. Thus

$$j = \begin{cases} \text{half-integer} & \Delta = \text{even}, \\ \text{integer} & \Delta = \text{odd}. \end{cases}$$

The representation D_j^+ is thus a 'true' unitary representation of so(2, 1) in this case and in this representation the eigenvalues of H_k are

$$E_k = 2\omega_k(n_k + \tfrac{1}{2} + \Delta_k/2) - (\omega_e + \omega_A), \qquad n_k = 0, 1, 2, 3, \ldots, \tag{54}$$

with

$$\omega_k^2 = (\omega_e + \omega_A)^2 - \omega_e^2\gamma_k^2. \tag{55}$$

The ferrimagnetic Hamiltonian is a generalisation of (42), where the spins of the atoms in the two sublattices have different values S_a and S_b. Supposing $S_a > S_b$, we can write

$$S_a = (1+\alpha)S, \qquad S_b = (1-\alpha)S, \qquad \alpha > 0. \tag{56}$$

In this case H_k assumes the form

$$H_k = \omega_e\gamma_k(S_aS_b)^{1/2}/S(C_k^+D_k^+ + C_kD_k) + (\omega_e + \omega_A)/S(S_aC_k^+C_k + S_bD_k^+D_k).$$

In terms of the operators (48)–(50),

$$H_k = -2\omega_e\gamma_k(1-\alpha^2)^{1/2}J_1 + 2(\omega_e + \omega_A)J_3 + \alpha(\omega_e + \omega_A)(C_k^+C_k - D_k^+D_k) - (\omega_e + \omega_A). \tag{57}$$

The term $C_k^+C_k - D_k^+D_k$ commutes with all the operators J_1, J_2, and J_3. It is in fact

related to the Casimir operator via (51) and (52). Hence H_k can be written as

$$H_k = -2\omega_e\gamma_k(1-\alpha^2)^{1/2}J_1 + 2(\omega_e+\omega_A)J_3 \pm \alpha(\omega_e+\omega_A)\Delta - (\omega_e+\omega_A). \tag{58}$$

The positive and negative signs in (58) correspond to the cases where the eigenvalues of C^+C are greater and less than those of D^+D respectively. The spectrum of H_k thus has two branches given by

$$\omega_k^{\pm} = [(\omega_e+\omega_A)^2 - \omega_e^2\gamma_k^2(1-\alpha^2)]^{1/2} \pm \alpha(\omega_e+\omega_A). \tag{59}$$

The effect of introducing an external magnetic field in the Hamiltonian (42) can be studied in a similar way.

5. Spin waves in a quadratic antiferromagnetic with easy-plane anisotropy

The two-dimensional antiferromagnets have generated considerable interest in recent years (Lebesque *et al* 1978, Balucani *et al* 1980, Lindgard and Kowalska 1976, Cooke and Lindgard 1977). In many of these antiferromagnets the anisotropy can be described by a staggered anisotropy field H_A (similar to that considered in the Hamiltonian (42)). However, some of these antiferromagnets have a crystal field anisotropy which favours the spins lying in a plane rather than being oriented along a preferential axis. Such an easy-plane anisotropy can be described by a term $D(S_{ix})^2$ with $D>0$ and X normal to the plane. The Hamiltonian in this case is obtained by adding this term for the two sublattices to the Hamiltonian (42) considered in § 4:

$$H = J\sum_{j\delta} \boldsymbol{S}_j\cdot\boldsymbol{S}_{j+\delta} - h\sum_j S_{jz}^{a} + h\sum_j S_{jz}^{b} + D\sum_j (S_{jx}^{a})^2 + D\sum_j (S_{jx}^{b})^2. \tag{60}$$

Detailed discussions of this Hamiltonian in the spin wave approximation have been provided by Lebesque *et al* (1978) and Balucani *et al* (1980).

Conventional techniques involving the Holstein-Primakoff transformation lead to an imaginary value of a uniform mode ($k=0$) energy (Lindgard and Kowalska 1976), hence modifications in the procedure for obtaining magnon variables are required. In terms of the lattice Fourier transforms of the boson operators referring to sites in sublattices a and b, the Hamiltonian (60) in the spin wave approximation assumes the form

$$H_{red} = \tfrac{1}{2}\sum_k H_k,$$

$$\begin{aligned} H_k = {} & A_1(a_k^+a_k + a_{-k}a_{-k}^+ + b_k^+b_k + b_{-k}b_{-k}^+) + A_2(a_k^+a_{-k}^+ + a_{-k}a_k + b_k^+b_{-k}^+ + b_{-k}b_k) \\ & + C_k(a_k^+b_k + a_{-k}b_{-k}^+ + b_k^+a_k + b_{-k}a_{-k}^+) \\ & + B_k(a_k^+b_{-k}^+ + a_{-k}b_k + b_k^+a_{-k}^+ + b_{-k}a_k). \end{aligned} \tag{61}$$

Here A_1 and A_2 depend on the parameters J, h and D as well as the expectation values of some quadratic and bilinear combinations of the boson operators which appear through 'mean field' or 'Hartree-Fock' type approximations. C_k and B_k depend, in addition to the above parameters, on the two-dimensional wavevector $\boldsymbol{k}$ in the YZ plane through a term

$$\gamma_k = (1/Z)\sum_{\delta} e^{i\boldsymbol{k}\cdot\boldsymbol{\delta}}$$

where δ connects a typical spin to its nearest neighbours in the plane.

Diagonalisation of a general quadratic boson Hamiltonian such as (61) has been discussed by various authors (Colpa 1978, Tikochinsky 1979, van Hemmen 1980). The spectrum of (61) consists of two separate branches of magnons with energies

$$E_k^{\pm} = 2[(A_1 \pm C_k)^2 - (A_2 \pm B_k)^2]^{1/2}. \tag{62}$$

These two branches can be shown to arise from two distinct so(2, 1) algebras. The form (62) of the magnon energy in the two branches suggests we write H_k as

$$H_k = 2\{[(A_1 + C_k)J_1 - (A_2 + B_k)J_2] + [(A_1 - C_k)L_1 - (A_2 - B_k)L_2]\}, \tag{63}$$

with

$$J_1 = (X + Y)/4, \qquad J_2 = -(X' + Y')/4, \tag{64}$$

$$L_1 = (X - Y)/4, \qquad L_2 = -(X' - Y')/4, \tag{65}$$

where

$$X = a_k^+ a_k + a_{-k} a_{-k}^+ + b_k^+ b_k + b_{-k} b_{-k}^+, \tag{66}$$

$$X' = a_k^+ a_{-k}^+ + a_{-k} a_k + b_k^+ b_{-k}^+ + b_{-k} b_k, \tag{67}$$

$$Y = a_k^+ b_k + a_{-k} b_{-k}^+ + b_k^+ a_k + b_{-k} a_{-k}^+, \tag{68}$$

$$Y' = a_k^+ b_{-k}^+ + a_{-k} b_k + b_k^+ a_{-k}^+ + b_{-k} a_k. \tag{69}$$

Writing $[J_1, J_2] = iJ_3$ and $[L_1, L_2] = iL_3$, we find

$$J_3 = \tfrac{1}{4}i[(a_k^+ a_{-k}^+ + b_k^+ b_{-k}^+ + a_k^+ b_{-k}^+ + b_k^+ a_{-k}^+) - (a_k a_{-k} + b_k b_{-k} + a_k b_{-k} + b_k a_{-k})], \tag{70}$$

$$L_3 = \tfrac{1}{4}i[(a_k^+ a_{-k}^+ + b_k^+ b_{-k}^+ + a_k b_{-k} + a_{-k} b_k) - (a_k a_{-k} + b_k b_{-k} + a_k^+ b_{-k}^+ + a_{-k}^+ b_k^+)]. \tag{71}$$

The Hermitian operators J_i and L_i $(i = 1, 2, 3)$ independently satisfy the commutation rules of so(2, 1) with J_1 and L_1 being the generators of the corresponding compact subalgebras so(2); and J_i commute with L_i:

$$[J_1, J_2] = iJ_3, \qquad [J_2, J_3] = -iJ_1, \qquad [J_3, J_1] = iJ_2,$$

$$[L_1, L_2] = iL_3, \qquad [L_2, L_3] = -iL_1, \qquad [L_3, L_1] = iL_2,$$

$$[J_i, L_j] = 0, \qquad i, j = 1, 2, 3.$$

Thus H_k is an element of the Lie algebra g_k, where $g_k \sim \mathrm{so}(2,1)_k^J \otimes \mathrm{so}(2,1)_k^L \sim \mathrm{su}(1,1)_k^J \otimes \mathrm{su}(1,1)_k^L \sim \mathrm{so}(2,2)_k$; and the total Hamiltonian is an element of the direct product algebra $\Pi_k \otimes \mathrm{so}(2,2)_k$. Diagonalisation of (63) can be achieved by separate rotations in the space of the two independent algebras $\mathrm{so}(2,1)_k^J$ and $\mathrm{so}(2,1)_k^L$, or by considering the two independent invariants of the rank-2 algebra $\mathrm{so}(2,2)_k$, with each invariant being of the form $\mathrm{Tr}\, M^2$ where

$$M = \frac{1}{2}\begin{bmatrix} (A_1 \pm C_k) & (A_2 \pm B_k) \\ -(A_2 \pm B_k) & -(A_1 \pm C_k) \end{bmatrix}. \tag{72}$$

The + and − signs in (72) refer to the parts of H_k that belong separately to the two nonoverlapping subalgebras so(2, 1) in the two-dimensional representation (39). Supposing $(A_1 \pm C_k) > (A_2 \pm B_k)$, (63) can be diagonalised to the form

$$H_k' = 2\{[(A_1 + C_k)^2 - (A_2 + B_k)^2]^{1/2} J_1 + [(A_1 - C_k)^2 - (A_2 - B_k)^2]^{1/2} J_2\},$$

reproducing the spectrum (62).

6. Spin waves in the XY model

Lieb *et al* (1961) have discussed two exactly soluble quantum mechanical models for an antiferromagnetic linear chain with nearest-neighbour interactions. These are the XY model and the Heisenberg-Ising model. The Hamiltonians for these models can be diagonalised exactly and are therefore worth discussing. The elementary excitations in these models obey the fermion statistics. Since the fermions have to obey the Pauli exclusion principle, the eigenspectrum of the occupation number operator is bounded and we need a compact algebra to generate this spectrum. We show that the compact counterpart of so(2, 1), i.e., so(3), serves as the dynamical algebra for these models. Solomon and Montogomery (1978) have discussed the generalised XY model and its relation to the two-dimensional Ising model. It was demonstrated that the compact algebra $\mathrm{so}(2N)\otimes\mathrm{so}(2N)$ is the SGA of the XY model of a cyclic lattice of N points. In the following we provide an equivalent description in terms of the compact algebra $\Pi_k \mathrm{so}(3)_k$.

The XY model consists of N spin-$\frac{1}{2}$'s (N even) arranged in a row and having only nearest-neighbour interactions. The Hamiltonian is written as

$$H=\sum_i [(1+\gamma)S_i^x S_{i+1}^x+(1-\gamma)S_i^y S_{i+1}^y], \tag{73}$$

where γ is a parameter characterising the degree of anisotropy in the XY plane. Because the Hamiltonian involves only the X and the Y components of the spin operators, it is called the XY model. As $\gamma\to 1$ it reduces to the Ising model.

Following Lieb *et al* (1961) we first transform (73) to a Hamiltonian of interacting fermions. The steps involved in this procedure are similar to those in the boson model. However, the fermion anticommutation rules do present some problems and additional steps are needed to achieve the goal. One first introduces the creation and annihilation (or the raising and lowering) operators referred to the site i in the chain:

$$a_i^+ = S_i^x + \mathrm{i}S_i^y, \qquad a_i = S_i^x - \mathrm{i}S_i^y, \tag{74}$$

in terms of which the spin operators are

$$\begin{aligned} &S_i^x=(a_i+a_i^+)/2, \qquad S_i^y=(a_i^+-a_i)/2\mathrm{i}, \\ &S_i^z=a_i^+a_i-\tfrac{1}{2}, \end{aligned} \tag{75}$$

and the Hamiltonian is

$$H=\tfrac{1}{2}\sum_i (a_i^+a_{i+1}+\gamma a_i^+a_{i+1}^+ + \mathrm{HC}). \tag{76}$$

These operators partly resemble Fermi operators:

$$\{a_i, a_i^+\}=1, \qquad a_ia_i=a_i^+a_i^+=0,$$

and partly Bose operators:

$$[a_i^+, a_j]=[a_i^+, a_j^+]=[a_i, a_j]=0, \qquad i\neq j.$$

To construct pure Fermi operators out of these we define

$$c_i=\exp\left(\pi\mathrm{i}\sum_1^{i-1} a_j^+a_j\right)a_i, \tag{77}$$

$$c_i^+ = a_i^+ \exp\left(-\pi i \sum_1^{i-1} a_j^+ a_j\right), \tag{78}$$

such that $c_i^+ c_i = a_i^+ a_i$, and the inverse transformation is

$$a_i = \exp\left(-\pi i \sum_1^{i-1} c_j^+ c_j\right) c_i, \qquad a_i^+ = c_i^+ \exp\left(\pi i \sum_1^{i-1} c_j^+ c_j\right). \tag{79}$$

The c's obey the Fermi statistics:

$$\{c_i, c_j^+\} = \delta_{ij}, \qquad \{c_i, c_j\} = \{c_i^+, c_j^+\} = 0. \tag{80}$$

For a cyclic chain the Hamiltonian is

$$H = \tfrac{1}{2} \sum_1^N [(c_i^+ c_{i+1} + \gamma c_i^+ c_{i+1}^+) + \text{HC}] + h, \tag{81}$$

where h is a correction term which can be ignored for large systems. To diagonalise $H(h=0)$ we transform to a new set of Fermi operators

$$c_k = \frac{1}{\sqrt{N}} \sum_j e^{-ikr_j} c_j, \qquad c_k^+ = \frac{1}{\sqrt{N}} \sum_j e^{ikr_j} c_j^+, \tag{82}$$

where r_j denote the positions of the spins on the chain. If a is the interspin distance, then

$$H = \sum_k H_k$$

$$H_k = \cos ka(c_k^+ c_k + c_{-k}^+ c_{-k})/2 + \gamma \sin ka\, i(c_k^+ c_{-k}^+ + c_k c_{-k})/2. \tag{83}$$

We now express H_k as an element of the angular momentum algebra $\text{so}(3) \sim \text{su}(2)$. We define Hermitian operators

$$J_1 = (c_k^+ c_k + c_{-k}^+ c_{-k} - 1)/2, \tag{84a}$$

$$J_2 = i(c_k^+ c_{-k}^+ + c_k c_{-k})/2, \tag{84b}$$

$$J_3 = (c_k^+ c_{-k}^+ - c_k c_{-k})/2, \tag{84c}$$

which close under the commutation of so(3):

$$[J_1, J_2] = iJ_3, \qquad [J_2, J_3] = iJ_1, \qquad [J_3, J_1] = iJ_2. \tag{85}$$

H_k in (83) is a linear combination of the generators J_1 and J_2. It can be diagonalised by considering a rotation about the axis J_3 and using the relations

$$e^{-iJ_3\theta} J_1 e^{iJ_3\theta} = J_1 \cos\theta + J_2 \sin\theta, \tag{86}$$

$$e^{-iJ_3\theta} J_2 e^{iJ_3\theta} = J_2 \cos\theta - J_1 \sin\theta, \tag{87}$$

or by realising J_i in terms of the Pauli spin matrices:

$$J_1 = \frac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \qquad J_2 = \frac{1}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix}, \qquad J_3 = \frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

and expressing H_k as a two-dimensional matrix:

$$M = \frac{1}{2}\begin{pmatrix} 0 & \cos ka - i\gamma \sin ka \\ \cos ka + i\gamma \sin ka & 0 \end{pmatrix}. \tag{88}$$

Tr M is a null invariant, while Tr M^2 generates the spectrum

$$E_k^2 = 1 - (1 - \gamma^2) \sin^2 ka. \tag{89}$$

The spectrum of the Hamiltonian in the Heisenberg-Ising model,

$$H = \sum_{i=1}^{N} \boldsymbol{S}_{2i-1} \cdot \boldsymbol{S}_{2i} + 2\gamma \sum_{i=1}^{N-1} S_{2i}^z S_{2i+1}^z,$$

can be obtained in a similar way.

7. Comments and conclusions

We have shown that the spin wave or the magnon energy spectrum in various localised-spin models can be obtained by using so(2, 1) ~ su(1, 1) or so(3) ~ su(2) as the dynamical algebra. In conventional localised-spin models (with nearest-neighbour exchange interaction as well as terms due to dipolar coupling, crystal field and planar anisotropies) of ferromagnetic, ferrimagnetic and antiferromagnetic Bravais lattices the Hamiltonian can be written in a quadratic boson form and can be studied by using the unitary representation of the noncompact algebra so(2, 1). The approximations involved in obtaining the quadratic boson form involve neglecting higher-order terms as well as replacing certain quadratic and bilinear combinations of boson operators by c-numbers. Such approximations are collectively referred to as 'spin wave approximations'. For the XY and Heisenberg-Ising models (Lieb *et al* 1961) of an antiferromagnetic linear chain no such approximation is necessary. In these exactly soluble models the Hamiltonian of the interacting spin system can be expressed as one of interacting fermions. These quadratic fermion Hamiltonians can be studied by using the unitary representation of the compact algebra so(3). Algebraic structure of quadratic boson and fermion Hamiltonians (Ui 1968, Holman and Biedenharn 1966, Moshinsky 1968) has been discussed previously in the literature (see appendix). In this paper we apply these results to systems of localised interacting spins.

It is desirable to extend this preliminary study to nonlocalised or itinerant spin models of magnetic systems and to describe one- and two-magnon bound states and soliton-like excitations in various models (Mattis 1981). Finally one would like to consider Hamiltonians with electron, phonon and magnon coordinates involving electron-phonon, electron-magnon and magnon-phonon couplings. Such a study is expected to shed some light on the interplay of superconductivity, magnetism and charge density waves (Huang 1985). Solomon and Birman (1984a, b, 1985) have already offered an algebraic description of a coexisting antiferromagnetic-superconducting system, for which the SGA is su(8).

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Appendix

A.1. so(3) *and the interacting fermion Hamiltonian*

Let us first consider an interacting many-fermion system. We suppose that this Hamiltonian can be diagonalised by transforming to a new set of fermion operators. In the diagonal form the Hamiltonian is

$$H = \sum_k H_k, \qquad H_k = \gamma_k a_k^+ a_k, \tag{A1}$$

where a_k^+ and a_k are the creation and destruction operators for the new set of noninteracting fermions in state k. In order to write H_k as an element of a Lie algebra, we define Hermitian operators

$$J_1 = \mathrm{i}(a_k^+ - a_k)/2, \qquad J_2 = (a_k + a_k^+)/2. \tag{A2}$$

The algebra generated by J_1 and J_2 closes upon introducing a third operator

$$J_3 = (a_k^+ a_k - \tfrac{1}{2}). \tag{A3}$$

The Hermitian operators J_1, J_2 and J_3 generate the algebra so(3) $\sim$ su(2). In a unitary representation of so(3) the eigenvalues and eigenstates of H_k are given by

$$H_k|j^k m^k\rangle = \gamma_k(J_3^k + \tfrac{1}{2})|j^k m^k\rangle = \gamma_k(m^k + \tfrac{1}{2})|j^k m^k\rangle, \tag{A4}$$

with $m^k = -j^k, -j^k+1, \ldots, j^k$. Here j^k is related to the eigenvalue of the Casimir operator or the total angular momentum

$$J_k^2 = (J_1^k)^2 + (J_2^k)^2 + (J_3^k)^2 \tag{A5}$$

by

$$J_k^2|j^k m^k\rangle = j^k(j^k+1)|j^k m^k\rangle. \tag{A6}$$

Using (A2) and (A3) we find

$$J_k^2 = \tfrac{3}{4}, \qquad j^k = \tfrac{1}{2}, -\tfrac{3}{2}. \tag{A7}$$

Thus the eigenstates of H_k provide a 'spinor' representation of so(3) $\sim$ su(2). Using the representation $D_{1/2}$ of so(3),

$$H_k|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \gamma_k|\tfrac{1}{2}\,\tfrac{1}{2}\rangle \tag{A8}$$

and

$$H_k|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle = 0. \tag{A9}$$

Thus the state $|\frac{1}{2}\,\frac{1}{2}\rangle$ corresponds to a filled k state, i.e.,

$$|\tfrac{1}{2}\,\tfrac{1}{2}\rangle \equiv a_k^+|0\rangle$$

where $|0\rangle$ is a state in which the kth level is empty.

Since the diagonalised Hamiltonian (A1) is an element of the algebra $\Pi_k \otimes \mathrm{so}(3)_k$, so must be the original Hamiltonian. The diagonalisation induces a rotation in the space of $\Pi_k \otimes \mathrm{so}(3)_k$ and brings the Hamiltonian into the form (A1). Thus we conclude that if an interacting fermion Hamiltonian can be transformed into the form (A1) then the SGA is so(3). It should be noted that the SGA of a general quadratic fermion Hamiltonian is larger than so(3) (Solomon and Birman 1982). It is only when the Hamiltonian can be diagonalised to the form (A1) that it can be described as an element of $\Pi_k \otimes \mathrm{so}(3)_k$.

For most real systems of fermions, the interactions give rise to terms higher than quadratic in the Hamiltonian. Since the diagonalisation requires a quadratic form, one has to resort to approximations. These are the familiar mean field, semiclassical or Hartree-Fock approximations where certain operators or combinations of operators in the Hamiltonian are replaced by classical numbers, in addition to dropping higher-order terms. In the theory of superconductivity such approximations give rise to the reduced BCS Hamiltonian. The SGA of this quadratic fermion Hamiltonian is known to be so(3) (Solomon (1981), see also spin analogue treatment of Anderson (1958)). In the XY and Heisenberg-Ising models of an antiferromagnetic linear chain no such approximation is necessary.

A.2. so(2, 1) *and the interacting boson Hamiltonian*

We now consider an interacting many-boson system whose Hamiltonian can be diagonalised and expressed in the form (A1). a_k^+ and a_k are now boson creation and annihilation operators referring to noninteracting bosons in state k. Upon introducing Hermitian operators

$$J_1 = (a_k^+ a_k^+ + a_k a_k)/4, \qquad J_2 = \mathrm{i}(a_k a_k - a_k^+ a_k^+)/4,$$
$$J_3 = (1 + 2a_k^+ a_k)/4, \tag{A10}$$

we realise the algebra so(2, 1) ~ su(1, 1), with J_3 being the generator of the compact subalgebra so(2) ~ su(1). The choice of generators (A10) is equivalent to the choice (13), (14) and (16) used in the discussion of the ferromagnetic Hamiltonian in terms of harmonic oscillator variables. The value of j in this case is $\frac{3}{4}$ or $\frac{1}{4}$. The 'one-boson' states split up into two representations $D^+_{3/4}$ and $D^+_{1/4}$ of so(2, 1), which are not its true representations. In order to discuss the boson Hamiltonian (A1) in true representations of so(2, 1) we can write it in the form

$$H = \sum_{k \geqslant 0} H_k = \sum_{k \geqslant 0} \gamma_k (a_k^+ a_k + a_{-k}^+ a_{-k}), \tag{A11}$$

assuming

$$\gamma_k = \gamma_{-k}.$$

Omitting for the time being the $k = 0$ (uniform mode) term in (A11) one can define Hermitian operators

$$J_1 = -\tfrac{1}{2}(a_k^+ a_{-k}^+ + a_k a_{-k}), \qquad J_2 = \tfrac{1}{2}\mathrm{i}(a_k^+ a_{-k}^+ - a_k a_{-k}),$$
$$J_3 = \tfrac{1}{2}(a_k^+ a_k + a_{-k}^+ a_{-k} + 1), \tag{A12}$$

which generate the algebra so(2, 1) with J_3 as the generator of the subalgebra so(2). The Casimir operator in this case has the value

$$j(1-j) = \tfrac{1}{4}(1 - \Delta^2), \qquad \Delta = |a_k^+ a_k - a_{-k}^+ a_{-k}|$$

and $j = (1 + \Delta)/2$, i.e., an integer (Δ = odd) or a half integer (Δ = even). In the absence of the $k = 0$ term, H_k can thus be expressed as a 'two-boson' Hamiltonian and realised in a 'true' representation of so(2, 1). For a superfluid boson system the Bogoliubov prescription for replacing the zero-momentum operators by their classical values, assuming macroscopic occupation of the zero momentum state, achieves this goal

(Solomon 1971). However if the $k = 0$ state is present, it needs to be treated separately by defining generators

$$J_1^0 = (a_0^+ a_0^+ + a_0 a_0)/4, \qquad J_2^0 = \mathrm{i}(a_0 a_0 - a_0^+ a_0^+)/4,$$
$$J_3^0 = (1 + 2a_0^+ a_0)/4,$$

of the algebra $\mathrm{so}(2, 1)_0$. Accordingly the $k = 0$ states of H_k are realised in the representations $D^+_{3/4}$ and $D^+_{1/4}$ of $\mathrm{so}(2, 1)_0$ and typify fractional angular momentum. Fortunately very often the $k = 0$ term does not give rise to interesting dynamical effects. For example the $k = 0$ phonon mode refers to uniform translation of the crystal and does not contribute to its transport properties. Hence this term can often be dropped or supposed to be included in the zero-point energy of the system without losing any interesting effects.

Thus we conclude that a quadratic boson Hamiltonian which can be put in the form (A5) can always be studied in unitary representations of so(2, 1). In order to achieve a realisation in a true representation it is necessary to write each H_k as a 'two-boson' Hamiltonian and treat H_0 separately. In the case of ferrimagnetic or antiferromagnetic solids, the 'two-sublattice' structure makes H_k a truly 'two-boson' Hamiltonian and there is no need to treat the $k = 0$ mode separately. It is to be noted that a general quadratic boson Hamiltonian may belong to an algebra larger than so(2, 1) (see e.g. realisation in symplectic algebras, Rowe (1978)). It is only when it can be transformed to the form (A1) that its SGA is $\Pi_k \mathrm{so}(2, 1)_k$.

References

Aldrovandi P and Leal Ferreira P 1969 *Lett. Nuovo Cimento* **1** 317

Anderson P W 1958 *Phys. Rev.* **112** 1900

Balucani U, Pini M G and Tognetti V 1980 *J. Phys. C: Solid State Phys.* **13** 2925

Barut A O 1964a *Phys. Rev.* **135** B 839

—— 1964b *On Dynamical Symmetry Groups and Mass Spectrum of Elementary Particles, Symmetry Principles at High Energy (Coral Gables Conference)* ed B Kursunoglu and A Perlmutter (San Francisco: Freeman) pp 81-92

—— 1973 *Siam J. Appl. Math.* **25** 247

Barut A O and Böhm A 1965 *Phys. Rev.* **139** B 1107

Barut A O and Phillips E C 1968 *Commun. Math. Phys.* **8** 52

Barut A O and Raczka R 1977 *Theory of Group Representations and Applications* (Warsaw: PWN)

Birman J L and Solomon A I 1982 *Phys. Rev. Lett.* **49** 230

Block F 1932 *Z. Phys.* **74** 295

Colpa J H P 1978 *Physica* **93A** 327

Cooke J F and Lindgard P-A 1977 *Phys. Rev.* B **16** 408

Cordero P and Ghirardi G C 1971 *Nuovo Cimento* **2A** 217

Cordero P and Hojman S 1970 *Lett. Nuovo Cimento* **4** 1123

Cordero P, Hojman S, Furlan P and Ghirardi G C 1971 *Nuovo Cimento* **3A** 807

Dothan Y, Gell-Mann M and Ne'eman Y 1965 *Phys. Lett.* **17** 148

Gottfried K 1974 *Quantum Mechanics* vol I (New York: Benjamin) p 259

Heller G and Kramers H A 1934 *Proc. R. Acad. Amsterdam* **37** 378

Herring C 1966 *Exchange Interactions Among Itinerant Electrons in Magnetism* ed G Rado and H Suhl (New York: Academic)

Holman W J and Biedenharn L C 1966 *Ann. Phys.* **39** 1

—— 1968 *Ann. Phys.* **47** 205

Holstein T and Primakoff H 1940 *Phys. Rev.* **58** 1098

Huang C Y 1985 to be published

Kittel C 1963 *Quantum Theory of Solids* (New York: John Wiley & Sons)

Lanik J 1967 *Nucl. Phys.* B **2** 263
—— 1968 *Nucl. Phys.* B **5** 523
Lebesque J V, Colpa J H P and Bevaart L 1978 *Physica* **95B** 123
Lieb E, Shultz T and Mattis D 1961 *Ann. Phys.* **16** 407
Lindgard P-A and Kowalska A 1976 *J. Phys. C: Solid State Phys.* **9** 2081
Mattis D C 1981 *The Theory of Magnetism I, Springer Series in Solid State Sciences* **17** (New York: Springer) ch 6
Moshinsky M 1968 *Group Theory and the Many Body Problem* (New York: Gordon and Breach) ch 1
Rowe D J 1978 *Can. J. Phys.* **56** 442
Slater J C 1930 *Phys. Rev.* **35** 509
Solomon A I 1971 *J. Math. Phys.* **12** 390
—— 1981 *J. Phys. A: Math. Gen.* **14** 2177
Solomon A I and Birman J L 1982 *Phys. Lett.* **88A** 413
—— 1984a *Proc. XIIIth Int. Colloq. on Group Theoretical Methods in Physics* 1984 *University of Maryland, College Park, MD, USA* ed W W Zachary (Washington, DC: Naval Research Laboratory)
—— 1984b *Phys. Lett.* **104A** 235
—— 1985 to be published
Solomon A I and Montgomery W 1978 *J. Phys. A: Math. Gen.* **11** 1633
Stearns M B 1978 *Phys. Today* (April) 34
Thorpe M F 1978 *Magnetic Excitations* in *Correlation Functions and Quasiparticle Interactions in Condensed Matter, NATO Advanced Study Institutes Series* vol 35, ed J Woods Halley (New York: Plenum)
Tikochinsky Y 1979 *J. Math. Phys.* **20** 406
Ui H 1968 *Ann. Phys.* **49** 69
van Hemmen J L 1980 *Z. Phys.* B **38** 271
Van Kranendonk J and Van Vleck J H 1958 *Rev. Mod. Phys.* **30** 1
Wybourne B G 1974 *Classical Groups for Physicists* (New York: John Wiley)